# Centres of mass by integration

Recall from earlier, that

**Fact** — The centre of mass of a system of particles with positions $\mathbf{x}_i$ and masses $m_i$ satisfies:

$$\overline{\mathbf{x}} \sum m_i = \sum m_i \mathbf{x}_i$$

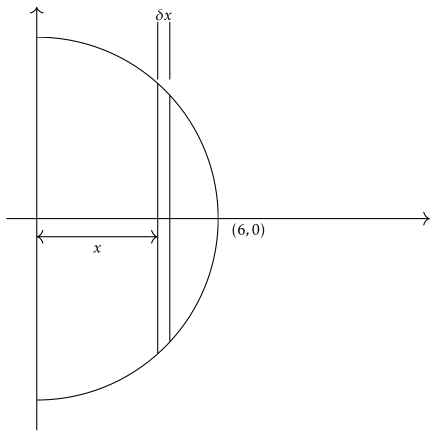
If we break down continuous shape into many small pieces, we these sums will tend to integrals, ie

**Fact** — The centre of mass of an object with *density* $m(\mathbf{x})$ at position $\mathbf{x}$, will have centre of mass $\overline{\mathbf{x}}$ satisfying:

$$\overline{\mathbf{x}} \int m(\mathbf{x})\, d\mathbf{x} = \int m(\mathbf{x})\mathbf{x}\, d\mathbf{x}$$

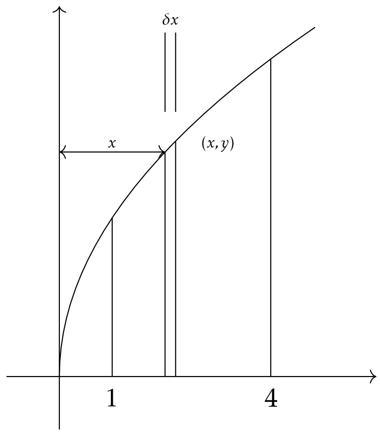
**Example**

A uniform semi-circular lamina has a radius of 6 cm and a mass per unit area $m$. Find the position of the center of mass.



**Example**

Find the centre of mass of a uniform lamina bounded by the curve $y^2 = 9x$, the $x$-axis and the coordinates $x = 1$ and $x = 4$, and lying in the first quadrant.



**Fact** — The coordinates of the centre of mass of a uniform lamina defined by the area between $y = f(x), x = 0, x = a$ and the $x$-axis are:

$$\overline{x}\int_0^a f(x)dx = \int_0^a xf(x)dx \qquad \overline{y}\int_0^a f(x)dx = \frac{1}{2}\int_0^a f(x)^2 dx$$

**Example**

A lamina has three straight edges along the lines $x = 0, x = 2, y = 0$.
The fourth edge is a curve modelled by $f(x) = x^2 + 4$.
Find the coordinates of the centre of mass.

**Example**

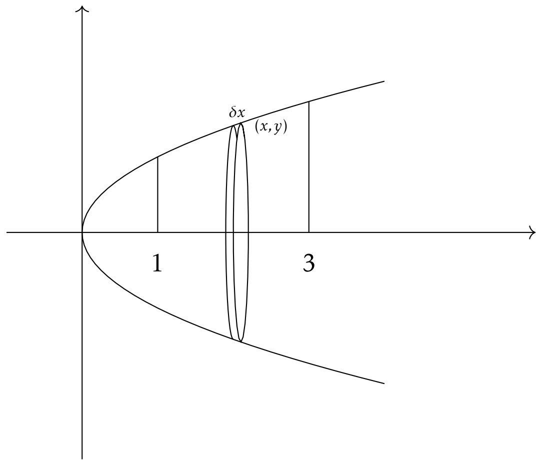
A *non-uniform* rod has length 8 m with mass density function $f(x) = 50 - x$ kgm$^{-1}$. Find the distance centre of of mass from the denser end.

## Solids of Revolution

By symmetry, the centre of mass must lie on the axis of rotation. The question then becomes, where do they lie on that axis?

**Example**

An area is enclosed by the curve $y^2 = 5x$, the $x$-axis, the lines $x = 1$, $x = 3$, and it lies in the first quadrant. The area is rotated about the $x$-axis through one revolution. Find the coordinates of the centre of mass of the uniform solid.

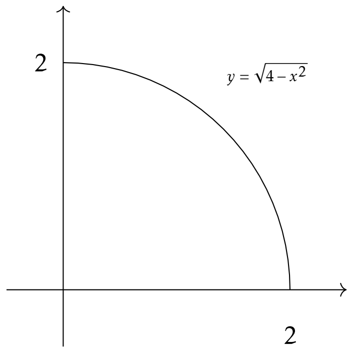


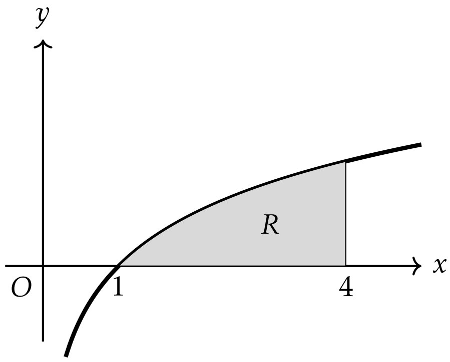
**Fact —** The centre of mass of a uniform solid of revolution with radius defined by $y = f(x)$ is:

$$\overline{x} = \frac{\displaystyle\int_0^a \pi x y^2 \, dx}{\displaystyle\int_0^a \pi y^2 \, dx}$$

**Example**

The region $R$ is bounded by the curve $y = \sqrt{4 - x^2}$ for $0 \leq x \leq 2$, the $x$-axis and the $y$-axis. $R$ is rotated by $2\pi$ radians about the $x$-axis to produce a solid of revolution. Find the centre of mass.






**Example** (OCR M4 2017 Q4)

The diagram shows the curve with equation $y = \frac{1}{2}\ln x$. The region $R$, shaded in the diagram, is bounded by the curve, the $x$-axis and the line $x = 4$. A uniform solid of revolution is formed by rotation $R$ completely about the $y$-axis to form a solid of volume $V$.

(i) Show that $V = \frac{1}{4}\pi(64\ln 2 - 15)$ **[4]**

(ii) Find the exact $y$-coordinate of the centre of mass of the solid **[7]**

**Example**

A body consists of a uniform solid circular cylinder $C$, together with a uniform solid hemisphere $H$ which is attached to $C$. The plane face of $H$ coincides with the upper plane face of $C$, as shown in the figure above. The cylinder $C$ has base radius $r$, height $h$ and mass $3M$. The mass of $H$ is $2M$. The point $O$ is the centre of the base of $C$.

(a) Show that the distance of the centre of mass of the body from $O$ is

$$\frac{14h + 3r}{20}.$$

**[5]**

The body is placed with its plane face on a rough plane which is inclined at an angle $\alpha$ to the horizontal, where $\tan\alpha = \frac{4}{3}$. The plane is sufficiently rough to prevent slipping. Given that the body is on the point of toppling,

(b) find $h$ in terms of $r$. **[4]**